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Researches and Evaluations in the Field of *MATHEMATICS*

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CHAPTER 1

MATHEMATICAL MODELS AND DIFFERENTIAL EQUATIONS

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1.Introduction

Mathematical models are invaluable in studying natural and artificial systems, helping scientists and engineers to predict and analyze complex phenomena. These models abstract real-world problems into mathematical language, enabling the derivation of insights that might be unattainable otherwise. Among the mathematical tools employed, differential equations stand out due to their ability to describe changes over time and space in various fields, such as physics, biology, economics, and chemistry. This chapter delves into the fundamentals of mathematical modeling, emphasizing the role of differential equations, and provides detailed examples illustrating their application ([1-5]).

When studying an event in nature or in different scientific fields, a structure that includes the more influential and important characteristics of the event is required. For example, for an architect, a model of the house to be built, for a mechanical engineer, the technical characteristics of the machine to be made, for an astronomer, the orbital paths of celestial bodies, and other similar characteristics are important. A structure that reflects the key characteristics of the object being studied, ignoring secondary characteristics that have little impact on the event, is called the model of the object. Modeling is the process of studying the real event on the model.

By creating a model, the real event is simplified, making it possible to study. While the model and the original are very close to each other, they are still different. Therefore, it is necessary to focus on both the similarities and differences between the model and the original. The model should not be exactly the same as the original; otherwise, it wouldn't be a model, but rather a reflection of the event in a mirror. However, the model must include the important characteristics of the original, otherwise, the differences would be too large, and it would not be possible to study the rules of the original with such a model.

The main advantage of a mathematical model over other types of models is that different events, which are completely distinct in their physical content, can be expressed by the same mathematical model. For example, there are different problems related to economics, sociology, and biology that can be studied using the same mathematical model.

When creating a mathematical model, the researcher must examine these events in depth and detail, identify their important parameters based on experimental results, and pay attention to the mutual laws (rules) between them.

Many studies show that trying to reflect a large number of real parameters or various characteristics of an object or event in the model makes solving the problem more difficult, rather than simplifying it. This often complicates the model to such an extent that it becomes impossible to work with.

Mathematical modeling is a transition towards abstracting special cases using symbols, or expressing them with various symbols. By using symbols, specific values of numbers can be represented. For example, by saying "b" is a decimal number, we can represent all numbers like 10, 100, 1000, etc. Symbols make it easier to express arithmetic operations (addition, subtraction, exponentiation, etc.). Now, let's consider the mutual relationship between symbols. For example, let the parameters of the related event be denoted by x and the others by y. If one of them, for example, changes with x, then the event can be expressed as

$$y = f(x).$$

When the relationship between the events changes like

$$x = 1,2,3,4,5,... \text{ and } y = 1,4,9,16,25,...,$$

the functional relationship between the events can be shown as

$$y = x^2.$$

Along with such simple relationships, more complex ones are also encountered. In such cases, the relationship between all (x_1, x_2, ..., x_n) and y events can be expressed as

$$y = f(x_1, x_2, ..., x_n)$$

after proper notations. Therefore, when studying any physical event, the fundamental laws of this event are expressed in mathematical form or formulas using symbols.

In theory and experiments, not only statistical events but also dynamic events, which change over time, are encountered. These kinds of events are usually expressed with differential equations. Equations that include the derivative of an unknown function are called differential equations. For example, when the unknown function is

$$y = y(x),$$

the equation

$$y' = e^{x}$$

is a differential equation, and it is clear that this equation satisfies the function

$$y = e^{x}.$$

When creating a model for a real event, the resulting differential equation is called the differential model of the real event. A differential model is a special case of the mathematical model. It should be noted that differential models have different types depending on the physical event. We will only discuss mathematical models expressed with differential equations that involve an unknown function depending on a single independent variable.

When creating a mathematical model, it was stated that a scientific law or rule related to the event is important. For example, in the mechanical branch of physics, we benefit from Newton's laws, in electricity from Kirchhoff's laws, and in studying the rate of chemical reactions, we use the law of mass action.

Sometimes, events occur in nature where no law is available to express the differential model. In this case, different assumptions (hypotheses) about the change in parameters are accepted after being examined through experiments. Not every differential model can be easily solved. Let's compare this with algebraic equations. Solutions of first and second-degree algebraic equations can be expressed with simple radicals (such as taking roots). While solutions to third and fourth-degree algebraic equations can also be expressed with radicals, the presence of many radicals in the formula makes the expression difficult. In equations of degree higher than four, it is not possible to express the solution with radicals

Although various methods are used to solve differential equations, the expressions obtained in the solution may not always be easy to analyze, meaning that the relationships between the parameters emphasized in models can be complex. In such cases, some information about the necessary properties of the solution may be sufficient. The properties of the solution to a differential equation are examined in the qualitative theory of differential equations.

To construct a mathematical model at any given time, a set of general conditions must be met. Some of these conditions are as follows:

1. The process must be universal, meaning it should evolve according to certain laws within a measurable period of time. The governing laws of the process do not change over time during the process being studied. For example, many processes in classical mechanics and electrical circuits can be viewed as universal processes.

2. The process must be describable with the help of a finite number of parameters mathematically. For example, in Newtonian mechanics, the free motion of a particle is determined uniquely by the coordinates $(x(t),\ y(t),\ z(t))$ at each time t and the velocity vector $(\dot{x}(t),\ \dot{y}(t),\ \dot{z}(t))$, meaning the particle's motion can be described mathematically with 6 parameters. Similarly, the motion of n particles can be described mathematically with $6n$ parameters.

3. The process must be describable using differentiable functions. For example, the orbit and velocity of a satellite orbiting the Earth can be expressed using differentiable functions, so this motion can be described by an ordinary differential equation. However, the Brownian motion of a particle, which traces a curve with an infinite number of zigzags within a finite time, cannot be expressed by a differentiable function. Therefore, it cannot be expected that this motion can be described by an ordinary differential equation.

4. The process must be deterministic. In other words, given the state of the process at any moment, the past and future should be uniquely determined. For example, if the coordinates and velocity of a cannonball at any point in its trajectory are known, its origin and impact point can be determined using the motion laws given by Newton. In contrast, if the temperature at all points in a medium is known at a particular time ($t = t_0$), it is not possible to uniquely determine the temperature of the medium (i.e., the event's past) at any time before ($t < t_0$).

There is no special method to create the differential model of an event. However, for many cases, the following general procedures need to be followed:

1. The conditions of the event are examined, and appropriate data and charts are prepared to understand the event.

2. An equation is created according to the laws governing the event.

3. The created equation is integrated, and the general solution of the equation is found.

4. The specific solution, consistent with the initially given conditions, is found.

5. If necessary, auxiliary parameters are found by utilizing additional conditions (for example, density coefficient, thermal conductivity coefficient, etc.)

6. A general rule describing the event is expressed in a formula, and the required values are found.

7. The results are examined and compared with experimental results.

2. Some Examples for Mathematical Models

Example 1 (The Importance of Advertising in Economics):

Advertising is done on the radio and television for the sale of product B. Through the exchange of information among consumers, more is learned about the product. After the advertisements, the rate of change in the number of consumers who know about product B is proportional to the number of people who know and don't know about the product. In this case, find the spread rate of the advertisement.

Solution:

Let $x(t)$ be the number of consumers who know about the product at time t. The rate of change of the number of consumers who know about the product is dx/dt. Since the promotion is proportional to this rate due to the relationships among consumers, the equation is:

$$dx/dt = k \cdot x \cdot (N - x)$$

Here, k is a positive constant of proportionality. This equation can be integrated to yield:

$$1/x \ln(N - x) = k \cdot t + c$$

Or, by letting $N \cdot c = c_1$ and $A = e^{c_1}$, we get:

$$x/(N - x) = A \cdot e^{(N \cdot k \cdot t)}$$

Therefore:

$$x = N/(1 + P \cdot e^{(-N \cdot k \cdot t)})$$

This equation is known as *the logistic curve* in economics.

Now, if we use the initial condition $x(0) = N$ in this equation, we get:

$$x = N/(1 + (\gamma - 1) \cdot e^{(-N \cdot k \cdot t)})$$

Example 2 (Physical):

Let the velocity of a particle moving along a straight line at time t be $v(t)$. Find the position function $S(t)$ of the particle, which shows its distance from the origin at time t.

Solution:

Let the straight line be the *Ox*-axis. If the distance from the origin at time t is $S(t)$, then since the derivative represents velocity, we have:

$$ds/dt = v(t)$$

Assuming that $v(t)$ is a continuous function, integrating gives the position function:

$$S(t) = \int v(t)dt + c$$

By giving different values to c, we get different position functions. To find the specific position function for a particle that is at a distance S_0 from the origin at time t_0, we set $c = S_0$.

Thus, the motion equation becomes:

$$S(t) = \int[t0 \text{ to } t]\ v(t)dt + S_0$$

This relation describes a specific motion.

Example 3 (Biology):

In an environment with sufficient nutrients, the growth rate of a bacterial colony is directly proportional to the number of bacteria in the environment. Find the growth rule of these bacteria.

Solution:

Let $x(t)$ be the number of bacteria in the environment at time t.

According to the problem's condition, the rate of change of the number of bacteria is proportional to the number of bacteria, so:

$$dx/dt = k \cdot x(t)$$

where $k > 0$ is a constant of proportionality that depends on the bacterial species.

From this, we get:

$$x(t) = c \cdot e\text{^}(k \cdot t)$$

To obtain a specific solution, we need to know the initial number of bacteria x_0 at time t_0. Substituting these values gives:

$$x(t) = x_0 \cdot e\hat{}(k(t - t_0))$$

This is an example of exponential growth.

Example 4 (Chemical):

In the chemical reaction between substances *A* and *B*, a substance *C* is formed. The rate of the reaction is proportional to the amount of the substances that have not yet reacted. Find the amount of substance *C* formed.

Solution: There are two cases to consider. In the first case, when substance A is converted into C, the rate of the reaction is proportional to the remaining amount of A. Thus:

$$dx/dt = k \cdot (a - x)$$

where $a - x$ represents the remaining amount of substance *A*, and $k > 0$ is the proportionality constant.

In the second case, when both substances *A* and *B* react to form *C*, the rate is proportional to the product of the remaining amounts of *A* and *B*. Thus:

$$dx/dt = k \cdot (a - x)(b - x)$$

where *a* and *b* are the initial amounts of substances *A* and *B*, respectively, and $k > 0$ is the proportionality constant.

Using the initial condition $x(0) = 0$, in the first case, the equation becomes:

$$dx/(a - x) = -k\,dt$$

Integrating gives:

$$x = a(1 - e^{\wedge}(-k \cdot t))$$

In the second case, solving yields:

$$x = (a \cdot b)/(b - a) \cdot (1 - e^{\wedge}(-k \cdot (b - a) \cdot t))$$

Example 5:

According to economic studies, the demand for essential consumer goods y is related to the demand for luxury goods z and income through the following relations:

$$y(x) = b1(x - a_1)/(x - c_1),\ x > a_1$$

$$z(x) = b2x(x - a2)/(x - c2),\ x > a_1,\ a_2 > a_1$$

Here, a_1 and a_2 represent income thresholds for purchasing the respective goods. As $x \to \infty$:

$$\lim x\to\infty,\ y(x) = b_1$$

$$\lim x\to\infty,\ z(x) = \infty$$

Thus, as income increases, the demand for essential goods becomes limited, while demand for luxury goods increases without limit.

Example 6:

Suppose the cost of a product *y* is related to its volume *x* by the equation:

$$y = 10x + 50$$

Find the marginal cost for producing 100 units of the product.

Solution:

The marginal cost is given by the derivative $y'(x)$. For x = 100, we find:

$$y'(100) = 10$$

Thus, producing one more unit when 100 units are produced costs 10 units of currency.

3. Applications and Implications

1. Predictive Analysis

Models allow for forecasting, such as predicting the spread of diseases using SIR models or estimating market trends.

2. Optimization

In engineering, differential equations help optimize processes, like minimizing energy usage or maximizing structural integrity.

3. Understanding Phenomena

By solving models, we gain insights into underlying mechanisms, like turbulence in fluids or oscillations in circuits.

4. Challenges in Mathematical Modeling

- **Complexity:** Real-world systems often involve numerous interacting variables, making models computationally intensive.
- **Nonlinearity:** Many systems exhibit nonlinear behavior, complicating analytical solutions.
- **Validation:** Experimental validation can be resource-intensive.

Despite these challenges, advances in computational power and numerical techniques have significantly enhanced the scope of mathematical modeling.

5. Conclusion

Mathematical modeling, particularly through differential equations, is a cornerstone of scientific inquiry. It bridges theoretical principles and real-world applications, enabling the analysis and prediction of complex systems. By refining models and leveraging computational tools, researchers can continue to unlock the mysteries of nature and design innovative solutions for modern challenges.

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CHAPTER 2

A NOTE ON SKEW-SYMMETRIC LIE ALGEBRAS OF UNCOUNTABLE DIMENSION

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1. INTRODUCTION

Given a field F with characteristic zero. By $M_n(F)$, we denote the algebra of all $n \times n$ matrices over F. This algebra turns into a Lie algebra structure with Lie product which is defined by $[U, W] = UW - WU$, where UW is the ordinary multiplication of the matrices U and W (for more details see [2, 3]), and it is denoted by $gl_n(F)$. Let $so_n(F)$ be a set of all skew-symmetric matrices. We say that a matrix U is skew-symmetric if it holds that $U^T = -U$, where U^T is the transpose of U.

For all $U, W \in so_n(F)$, we have $[U, W] \in so_n(F)$, that is,

$$\begin{aligned}[U,W]^T &= (UW - WU)^T \\ &= (UW)^T - (WU)^T \\ &= W^T U^T - U^T W^T \text{ (since } W^T \\ &= -W, U^T = -U) \\ &= WU - UW \\ &= [W, U] \text{ (anti} - \text{commutativity)} \\ &= -[U, W].\end{aligned}$$

Thus, $so_n(F)$ is a subalgebra of $gl_n(F)$. Under natural embeddings $so_n(F) \to so_{n+1}(F)$ defined by

$$U \to \begin{pmatrix} U & 0 \\ 0 & 0 \end{pmatrix}$$

the direct limit $so_\infty(F)$ of $so_n(F)$ is a countably dimensional simple Lie algebra. It is straightforward to verify that this is a Lie algebra consisting of infinite $\mathbb{N} \times \mathbb{N}$ matrices U satisfying skew-symmetric property.

In this work, our purpose is to define a new uncountably dimensional Lie algebra occurring with infinite matrices and to show that this Lie algebra is simple. For this firstly, we denote a set of all infinite $\mathbb{N} \times \mathbb{N}$ matrices over F which have just finite number of non-zero rows by $M(\infty, F)$. Recall that the addition of two matrices, the matrix multiplication of two matrices and also the multiplication of a matrix with a scalar in F are well-defined. Thus, $M(\infty, F)$ is an associative F-algebra. Hence, the F-algebra $M(\infty, F)$ becomes a Lie algebra with the product $[U, W] = UW - WU$ and it is denoted by $gl(\infty, F)$. A basis of the algebra $gl(\infty, F)$ consists of the matrix units u_{ij} for $i, j \in \mathbb{N}$ which is the infinite matrix whose (ij)-th entry is 1 and all other entries are 0. The product of u_{ij} and u_{kl} is given by following rule

$$[u_{ij}, u_{kl}] = \delta_{jk} u_{il} - u_{li} u_{kj} \tag{1}$$

where δ_{ij} is the Kronecker delta function which is defined by

$$\delta_{ij} = \{\begin{matrix} 1, & \text{if } i = j \\ 0, & \text{otherwise} \end{matrix}\}$$

(see [1]).

We denote a Lie subalgebra of $gl(\infty, F)$ having all skew-symmetric matrices U by $so(\infty, F)$. Recall that the set $\{u_{ij} \mid i, j \in \mathbb{N}\}$ forms a generating set for $gl_n(F)$ and the set $\{u_{ij} - u_{ji} \mid i, j \in \mathbb{N}, i < j\}$ forms a basis of $so_n(F)$. Therefore, $gl_n(F)$ and $so_n(F)$ are countably dimensional. Obviously, $gl(\infty, F)$ and $so(\infty, F)$ are uncountably dimensional. Arbitrary element in $gl(\infty, F)$ is spanned by the set $\{u_{ij} \mid i, j \in \mathbb{N}\}$. Similarly, the set $\{u_{ij} - u_{ji} \mid i, j \in \mathbb{N}, i < j\}$ spans $so(\infty, F)$. We use the symbol 0 for the matrix whose entries are all zero.

2. MAIN RESULT

Now, we are prepared to give our main theorem.

Theorem 2.1. $so(\infty, F)$ is an uncountably dimensional simple Lie algebra.

Proof. Consider that K is a non-zero ideal of $so(\infty, F)$. First of all, we remark that it is sufficient to show that for any $i < j$, we have $u_{ij} - u_{ji} \in K$. Suppose that $W \in K$ is a non-zero matrix as the following

$$W = c_{12}(u_{12} - u_{21}) + c_{23}(u_{23} - u_{32}) + \cdots + c_{n-1,n}(u_{n-1,n} - u_{n,n-1})$$

and $c_{k-1,k} \neq 0 \quad (1 < k \leq n)$. Here, it is clear to observe that

$$u_{i,i+1} - u_{i+1,i} \in S \text{ for } 1 \leq i \leq n-1.$$

By using the product (1), we infer that

$$\begin{aligned}[W, u_{k,n+1} - u_{n+1,k}] &= [c_{12}(u_{12} - u_{21}) + c_{23}(u_{23} - u_{32}) + \cdots \\ &\quad + c_{n-1,n}(u_{n-1,n} - u_{n,n-1}), u_{k,n+1} - u_{n+1,k}] \\ &= c_{k-1,k}[u_{k-1,k} - u_{k,k-1}, u_{k,n+1} - u_{n+1,k}] \\ &= c_{k-1,k}(u_{k-1,n+1} - u_{n+1,k-1}) \in K\end{aligned}$$

and so $u_{k-1,n+1} - u_{n+1,k-1}$ is in K. Moreover, for each $p \geq 1$, we obtain

$$\begin{aligned}[W, u_{k,n+p} - u_{n+p,k}] &= [c_{12}(u_{12} - u_{21}) + c_{23}(u_{23} - u_{32}) + \cdots \\ &\quad + c_{n-1,n}(u_{n-1,n} - u_{n,n-1}), u_{k,n+p} - u_{n+p,k}] \\ &= c_{k-1,k}[u_{k-1,k} - u_{k,k-1}, u_{k,n+p} - u_{n+p,k}] \\ &= c_{k-1,k}(u_{k-1,n+p} - u_{n+p,k-1}) \in K\end{aligned}$$

and hence $u_{k-1,n+p} - u_{n+p,k-1} \in K$ for each $p \geq 1$.

Now we focus on the case that $i \neq k$ and $1 \leq i \leq n$, by doing a straightforward

calculation we have

$$\begin{aligned}&[W, u_{ki} - u_{ik}] \\ &= [c_{12}(u_{12} - u_{21}) + c_{23}(u_{23} - u_{32}) + \cdots \\ &\quad + c_{n-1,n}(u_{n-1,n} - u_{n,n-1}), u_{ki} - u_{ik}] \\ &= c_{k-1,k}(u_{k-1,i} - u_{i,k-1}) \in K.\end{aligned}$$

This means that $u_{k-1,i} - u_{i,k-1} \in K$. Consequently, we observe that all generators of $so(\infty, F)$ belong to K. This completes the proof.

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